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# Complexes of Differential Systems*

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The differential system:

$$dy_i/dx = f_i(x, y_1, \ldots, y_n), \qquad (i = 1, 2, \ldots, n), \tag{1}$$

where each $f_i$ is defined, real-valued, and continuous in a domain $D$ of $(n + 1)$-dimensional real space is considered. If $P(c, c_1, \ldots, c_n)$ is an interior point of $D$, there will exist at least one set of absolutely continuous functions: $y_1(x), \ldots, y_n(x)$ on an interval $I_p : c - d < x < c + d$, $d > 0$, which satisfies system (1) almost everywhere on $I_p$ and has

$$y_i(c) = c_i, \qquad (i = 1, \ldots, n). \tag{2}$$

If additional hypotheses, such as a Lipschitz condition or existence of continuous first partial derivatives with respect to the $y_i$'s are made for the functions $f_i$, then the solution of system (1), (2) is unique. This paper is concerned with systems where no such hypotheses are made and, hence, uniqueness may fail.

A point $P$ of $D$ for which system (1), (2) has more than one solution is called a *nonunique point*. Hamilton [1] has shown that the set of nonunique points in $D$ consists of an at most countable collection of closed sets.

By the *solution cone* through $P$, $T_p$, is meant the subset of $D$ containing those points, and only those points, which belong to solutions of system (1), (2). Since our work will concern a single "vertex" point $P$, $T_p$ will be replaced by the simpler notation $T$. The notation $T(d, e)$ is used to designate the "frustrum" of the cone $T$ which is determined by $c \leqslant d \leqslant x \leqslant e$. In particular, $T(e, e)$ is shortened to $T(e)$ and this designates the intersection of the cone $T$ with the hyperplane $x = e$.

The following theorem is implicit in many proofs of the Peano existence theorem [2] but is included here, together with a brief proof, to feature

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the fact that a single interval of positive length can be found on which all solutions of the cone through $P$ exist.

THEOREM 1. *There exists a positive number, $b$, such that any solution of* (1), (2) *exists and lies in $D$ for all $x$ on $[c - b, c + b]$.*

*Proof.* Since $P$ is interior to $D$, its distance from the boundary of $D$ is positive. Choose $r > 0$ but less than this distance and let $B$ be a closed spherical ball with center at $P$ and radius $r$. The continuous function $ds/dx = \sqrt{1 + f_1^2 + f_2^2 + \cdots + f_n^2}$ is bounded on $B$. Let $b = r/N$, where $N$ is the least upper bound for $ds/dx$ on $B$. Arc length along any solution curve through $P$ and lying entirely in $B$ is

$$\int_c^x (ds/dt)\, dt = \int_c^x [1 + f_1^2 + \cdots + f_n^2]^{1/2}\, ds \leqslant N(x - c).$$

Since $r$ is the minimum distance from $P$ to the boundary of $B$, it follows that $N \mid (x_q - c) \mid \geqslant r$ for any point $Q(x_q, q_1, \ldots, q_n)$ on the boundary of $B$. Hence all points of the solution curve for which $\mid x - c \mid \leqslant b = r/N$ lie in or on the boundary of $B$ and thus lie in $D$.

Theorem 1 shows that $T(d, e)$, where $c \leqslant d \leqslant e \leqslant c + d$, is bounded. Kamke [7] gave an example of a second-order system where the boundary of the solution cone contains continuous curves which are not solutions of the differential system. Theorem 2 shows that each point of a curve of this type belongs to a solution of the given system; hence $T(d, e)$ contains all of its boundary points. The proof given for this theorem follows a well-known procedure for establishing existence theorems through use of equicontinuous families of functions and the Ascoli theorem.

THEOREM 2. *The cone $T(d, e)$, where $d$ and $e$ belong to $[c, c + b]$, is closed.*

*Proof.* Let $A(a, a_1, \ldots, a_n)$ be a limit point of $T(d, e)$ and let

$$A_j(a_{0j}, a_{1j}, \ldots, a_{nj}), \qquad j = 1, 2, \ldots,$$

be a sequence of points of $T(d, e)$ which has $A$ as its sequential limit point. For each $j$, there is a solution: $y_{1j}(x), \ldots, y_{nj}(x)$ of (1), (2) such that $y_{ij}(a_{0j}) = a_{ij}$, $i = 1, \ldots, n$. The families of functions $\{y_{1j}(x)\}, \ldots, \{y_{nj}(x)\}$, $j = 1, 2, \ldots$, are equicontinuous on $[c, c + b]$, since each of the functions has a derivative that is bounded by $N$ on this interval and application of the mean value theorem gives $\mid y_{ij}(u) - y_{ij}(v) \mid \leqslant N \mid u - v \mid$ for any two points $u, v$ of $[c, c + b]$. Using Ascoli's theorem, a subsequence from $\{y_{1j}\}$ can be selected which converges uniformly on $[c, c + b]$ to a limit function $y_1(x)$. Applying Ascoli's theorem to the subsequence of $\{y_{2j}\}$, which has second subscripts corresponding to those in the sequence that gave $y_1(x)$, yields a subsequence of this sequence which converges uniformly to a function $y_2(x)$. Continuation of this process yields functions $y_1(x),..., y_n(x)$ which give a continuous curve in $D$ that is the uniform limit on $[c, c + b]$ of a sequence of solution curves for system (1), (2). Continuity of $f_1 ,..., f_n$ then shows that $y_1(x),..., y_n(x)$ is a solution of the differential system which contains $A$ and $P$. The point $A$ belongs to $T(d, e)$.

Kneser [3] has shown that the set $T(e) = T(e, e)$ is a continuum, hence closed and connected. Closure of this set also follows as a corollary of Theorem 2, above. Use of Kneser's theorem and Theorems 1 and 2, establishes:

THEOREM 3. *For $d$ and $e$ on $[c, c + b]$, the set $T(d, e)$ is a continuum.*

Osgood [4], Montel [5], Perron [6], and others have shown the existence of a maximal solution and a minimal solution for system (1), (2) in the case $n = 1$. If $U(x)$ and $u(x)$, respectively, denote these maximal and minimal solutions, then any other solution, $y(x)$, would satisfy $u(x) \leqslant y(x) \leqslant U(x)$ on $[c, c + b]$. Kamke [7] gave an example of a second-order system for which no maximal or minimal solution exists. Kamke then developed conditions on the functions $f_1(x, y_1 ,..., y_n)$ of system (1) that would insure existence of such solutions. By a maximal solution of the $n$-th order system (1), (2) we understand a solution $U_1(x),..., U_n(x)$ such that if $y_1(x),..., y_n(x)$ is any solution of the system, then $y_i(x) \leqslant U_i(x)$, $i = 1,..., n$. The minimal solution $u_1(x),..., u_n(x)$ is defined in a similar manner, with $u_i(x) \leqslant y_i(x)$ holding for each $i$ and all $x$ on the specified interval.

L. P. Burton and W. M. Whyburn [8] extended the work cited above to cover cases where solutions of maximal and minimal types might fail to exist but solutions of minimax types would exist.

DEFINITION. A solution of system (1), (2) is of $(k)$ max $-$ $(n - k)$ min *type*, abbreviated *type $k$*, if $k$ of the $y$'s are maximal while the remaining $y$'s are minimal. Thus, the solution $U_1(x),..., U_k(x)$, $u_{k+1}(x),..., u_n(x)$ is a $(k)$ max $-$ $(n - k)$ min solution if $U_i(x) \geqslant y_i(x)$ for $i = 1,..., k$ while $y_j(x) \geqslant u_j(x)$ for $j = k + 1,..., n$, where $y_1(x),..., y_n(x)$ is any solution of system (1), (2). A solution of (1), (2) is called a *critical solution* if it is of type $k$ for some $k$ in the set: 0, 1,..., $n$.

It is noted that $k = n$ yields the maximal solution, while $k = 0$ gives a minimal solution. The principal condition on $f_1 ,..., f_n$ which insures existence of a $(k)$ max $-$ $(n - k)$ min solution on $[c, c + b]$ for (1), (2) is that it be possible to permute subscripts on $y_1 ,..., y_n , f_1 ,..., f_n$ in such a way that $f_i(x, y_1 ,..., y_n)$, $i \leqslant k$, is monotonic increasing in $(y_1 ,..., y_k)'$ and monotonic decreasing in $y_{k+1}, \ldots, y_n$ while $f_j(x, y_1, \ldots, y_n)$, $j > k$, is monotonic decreasing in $y_1, \ldots, y_k$ and monotonic increasing in $(y_{k+1}, \ldots, y_n)'$. The prime indicates that the condition need not apply to $y_i$ in $f_i$ or to $y_j$ in $f_j$.

The work of Burton and Whyburn has been extended to infinite systems by V. Lakshmikantham and S. Leela [9].

When solutions of type $k$ exist, they lie entirely in the boundary of the solution cone, $T(e)$, and may be approximated uniformly by $(k)$ over — $(n - k)$ under functions as described by Burton and Whyburn [8]. Solutions other than these critical ones cannot be approximated in this manner. These properties are especially useful in applied problems where a solution must be selected from the cone of solutions and this solution must be approximated uniformly on the given interval. Due to some malfunction in a physical system, a nonunique point for the simulating differential system may be reached so that a selection of the type just described is required. Since critical solutions do not exist in all cases where uniqueness fails, enlargement of the class of differential systems which do have such solutions is highly important. The principal results of this paper show that any differential system (1), (2) which has a critical solution of type $k$, where $k$ is one of the numbers $0, 1, 2, \ldots, n$, can be used to identify other differential systems which have critical solutions through the point $P$ of Eq. (2). Through this, a complex of differential systems is associated with each nonunique point where a critical solution exists for one system. If $M_k$ denotes the number of systems in the complex with critical solutions of type $k$, $(k = 0, 1, \ldots, n)$, relations between these numbers are found which are similar to those found by Marston Morse [10] for critical points of functions of $n$ real variables.

THEOREM 4. *Let $P$ be a nonunique point for system* (1), (2) *and let there be an index $k$ such that for $x \geqslant c$:*

*(1) $f_r$ is monotonic increasing in $y_i$ and monotonic decreasing in $y_j$, where $(r = 1, \ldots, k)$; $(i = 1, \ldots, k)$; $(i \neq r)$; $(j = k + 1, \ldots, n)$.*

*(2) $f_s$ is monotonic decreasing in $y_i$ and monotonic increasing in $y_j$, where $(s = k + 1, \ldots, n)$; $(j = k + 1, \ldots, n)$; $(j \neq s)$; $(i = 1, \ldots, k)$. Let $z : [z_1(y_1), z_2(y_2), \ldots, z_n(y_n)]$ be a $n$-vector with components which are continuous and monotonic decreasing in a neighborhood of $(c_1, c_2, \ldots, c_n)$. Finally, let $z_i(c_i) = c_i$ for each $i = 1, 2, \ldots, n$. Then system* (3), (2) *has critical solutions*[1] *of maximal and minimal types on $[c, c + b]$, where*

$$
\begin{aligned}
y_i' &= f_i(x, y_1, \ldots, y_k, z_{k+1}, \ldots, z_n) = F_i(x, y_1, \ldots, y_n), \quad (i = 1, \ldots, k), \\
y_j' &= f_j(x, z_1, \ldots, z_k, y_{k+1}, \ldots, y_n) = F_j(x, y_1, \ldots, y_n), \quad (j = k + 1, \ldots, n).
\end{aligned} \tag{3}
$$

[1] Which, for some $z(x)$, may fail to be distinct.

*Proof.* Since $f_i$ is monotonic increasing in $y_1, ..., y_k$ and monotonic decreasing in $y_{k+1}, ..., y_n$ while each $z_i$ is monotonic decreasing, it follows that $F_i$ is monotonic increasing in $y_1, ..., y_n$ for $i = 1, ..., k$. Similarly, $F_j$ is monotonic increasing in $y_1, ..., y_n$ for $j = k + 1, ..., n$. As always, monotonicity of $f_r$ or $F_r$ in $y_r$ is not required. System (2), (3) satisfies the hypotheses of Kamke's [7] theorem, also case $k = n$ of Burton and Whyburn's [8] Theorem 4, and hence has maximal and minimal solutions on $[c, c + b]$.

Let $N$ denote the set $\{0, 1, ..., n\}$ and choose $h$ as an element of this set. Let $r : \{r_1, ..., r_h\}$ be $h$ distinct elements of $N$ and let $r'$ denote the compliment of $r$ with respect of $N$. The differential system

$$y_i' = g_i(x, y_1, ..., y_n), \qquad i = 1, ..., n, \tag{$3_{hr}$}$$

is formed from (3) as follows:

When $i \in r$: In $F_i$ replace $y_j$ by $z_j$ if $j \in r'$. Otherwise, leave $y_j$ unchanged. The resulting function is $g_i$.

When $i \in r'$: Replace $y_j$ in $F_i$ by $z_j$ if $j \in r$. Otherwise, leave $y_j$ unchanged. The resulting function is $g_i$.

THEOREM 5. *Under the hypotheses of Theorem* 4, *system* $(3_{hr})$, (2) *has critical solutions of types $h$ and $(n - h)$, both of which may degenerate into the unique solution.*

*Proof.* This is an immediate consequence of Theorem 4 of Burton and Whyburn [8] when it noted that system $(3_{hr})$ can be rearranged so that the numbers of set $r$ assume the roles of $1, 2, ..., h$ in the theorem cited.

For each $h$, system $(3_{hr})$ can be formed in ${}_nC_h = n!/h!(n - h)!$ ways, although not all of these systems may differ from each other. The notation $C_h = {}_nC_h$ is used to denote the number of combinations of $h$ elements in a set of $n$ elements.

THEOREM 6. *If $m = [(n + 1)/2]$ is the largest integer less than or equal to $(n + 1)/2$, then the inequalities $C_{h-1} \leqslant C_h$ are valid for $h = 1, 2, ..., m$.*

*Proof.*

$$C_h = \frac{n!}{h!(n-h)!} = \frac{n!}{(h-1)!(n-h+1)!}\,\frac{n-h+1}{h} = \frac{n-h+1}{h}\,C_{h-1}$$

and the desired inequality holds if $n - h + 1 \leqslant h$, or $h \leqslant (n + 1)/2$.

THEOREM 7. *The following inequalities are valid for* $m = [(n+1)/2]$:

$$
\begin{aligned}
&1 \leqslant C_0\,,\\
&1 \geqslant C_0 - C_1 + 1,\\
&1 \leqslant C_0 - C_1 + C_2\,,\\
&\ \vdots \qquad \vdots \qquad \vdots\\
&1 \leqslant [C_0 - C_1 + \cdots + (-1)^m C_m + \{1-(-1)^m\}/2](-1)^m,\\
&1 \leqslant [C_n - C_{n-1} + \cdots + (-1)^m C_{n-m} + \{1-(-1)^m\}/2](-1)^m,\\
&1 \leqslant [C_n - C_{n-1} + \cdots + (-1)^{m-1} C_{n-m+1} + \{1-(-1)^{m-1}\}/2](-1)^{m-1},\\
&\ \vdots\\
&1 \geqslant C_n - C_{n-1} + 1,\\
&1 \leqslant C_n\,,\\
&0 = C_0 - C_1 + \cdots + (-1)^n C_n\,.
\end{aligned}
$$

*Proof.* The last equation results from setting $x = 1$ in the binomial expansion of $(1-x)^n$. For $h \leqslant m$ and odd, the inequality has the form $1 \geqslant 1 + (C_0 - C_1) + (C_2 - C_3) + \cdots + (C_{h-1} - C_h)$ and Theorem 6 shows that the quantity in each parenthesis is less than or equal to zero. For $h$ even and $h \leqslant m$, the inequality has the form

$$1 \leqslant C_0 + (C_2 - C_1) + \cdots + (C_h - C_{h-1}),$$

where $C_0 = 1$ and the number in each parenthesis is nonnegative. Hence the first $m + 1$ of the inequalities are established. The remaining inequalities are obtained from these by replacing $C_k$ by $C_{n-k}$ in them.

For a given function $z : [z_1(y_1),..., z_n(y_n)]$, let $H(z)$ be the collection of different systems $(3_{hr})$ obtained when $h$ and $r$ range over $N$ and its subsets. $H(z)$ is called the *z-complex* for system (1), (2). Clearly, each system in $H(z)$ will occur at least two times, the second occurrence arising when the roles of $r$ and $r'$ are interchanged. For each $h$, $h \leqslant [(n+1)/2]$, let $M_h$ denote the number of different differential systems in $H(z)$, each of which has critical solutions of type $h$. For $h > [(n+1)/2]$, let $M_h = M_{n-h}$. Let $R_h = C_h - M_h + 1$, for $h = 0,..., n$.

THEOREM 8. *The following inequalities are valid, where* $m = [(n+1)/2]$.

$$
\begin{aligned}
&1 \leqslant M_0 + R_0 - 1,\\
&1 \geqslant (M_0 + R_0 - 1) - (M_1 + R_1 - 1) + 1,\\
&1 \leqslant (M_0 + R_0 - 1) - (M_1 + R_1 - 1) + (M_2 + R_2 - 1),\\
&\ \vdots
\end{aligned}
$$

$$
\begin{aligned}
1 &\leqslant [(M_0 + R_0 - 1) + \cdots + (-1)^m(M_m + R_m - 1) \\
&\quad + \tfrac{1}{2}\{1 - (-1)^m\}](-1)^m, \\
1 &\leqslant [(M_n + R_n - 1) + \cdots + (-1)^m(M_{n-m} + R_{n-m} - 1) \\
&\quad + \tfrac{1}{2}\{1 - (-1)^m\}](-1)^m, \\
1 &\leqslant [(M_n + R_n - 1) + \cdots + (-1)^{m-1}(M_{n-m+1} + R_{n-m+1} - 1) \\
&\quad + \tfrac{1}{2}\{1 - (-1)^{m-1}\}](-1)^{m-1}, \\
&\vdots \\
1 &\geqslant (M_n + R_n - 1) - (M_{n-1} + R_{n-1} - 1) + 1, \\
1 &\leqslant M_n + R_n - 1, \\
0 &= (M_0 + R_0 - 1) - (M_1 + R_1 - 1) + \cdots + (-1)^n(M_n + R_n - 1).
\end{aligned}
$$

*Proof.* These relations follow from Theorem 7 when $C_k$ is replaced by $M_k + R_k - 1$, $(k = 0,..., n)$, in the inequalities of that theorem.

COROLLARY 1. *The inequalities of the theorem are valid if the terms* $\frac{1}{2}\{1 - (-1)^k\}$, $(k = 1, 3,...)$, *are omitted.*

COROLLARY 2. *In the special case,* $R_k = 1$ *for* $(k = 0, 1,..., n)$, *the inequalities in Corollary* 1 *become*

$$
\begin{aligned}
1 &\leqslant M_0\,, \\
1 &\geqslant M_0 - M_1\,, \\
&\vdots \\
1 &\leqslant [M_0 - M_1 + \cdots + (-1)^m M_m](-1)^m, \\
1 &\leqslant [M_n - M_{n-1} + \cdots + (-1)^m M_{n-m}](-1)^m, \\
&\vdots \\
1 &\leqslant M_n\,, \\
0 &= M_0 - M_1 + M_2 + \cdots + (-1)^n M_n\,.
\end{aligned}
$$

The inequalities given in Theorem 8 and its corollaries are analogous to those obtained by Morse [11], the principal differences being that our $R_{k-1}$ is replaced by its negative in Morse's inequalities while Morse's final equation has 1 where our equation has zero. No doubt, some of these differences would be removed if our differential systems were given boundary conditions which are analogous to those used by Morse. The case covered by Corollary 2 of Theorem 8 is analogous to Morse's case where the domain is an $(n - 1)$-sphere and its interior.

EXAMPLE. Let the given differential system consist of the origin and

$y_1' = -7xy_2$, $y_2' = 5xy_3$, $y_3' = -3xy^{1/7}$. For $x \geqslant 0$, this system has the (1) max — (2) min solution: $U_1 = x^7$, $u_2 = -x^5$, $u_3 = -x^3$ and the (1) min — (2) max solution: $u_1 = -x^7$, $U_2 = x^5$, $U_3 = x^3$. Another solution is the identically vanishing one and it fits, appropriately, between these critical ones. If $z$ is taken as $[-y_1, -y_2, -y_3]$ and Theorem 4 applied, there results $y_1' = 7xy_2$, $y_2' = 5xy_3$, $y_3' = 3xy^{1/7}$. This system has the maximal solution: $U_1 = x^7$, $U_2 = x^5$, $U_3 = x^3$ and the corresponding minimal solution: $u_1 = -x^7$, $u_2 = -x^5$, $u_3 = -x^3$. Application of Theorem 5 to this system yields explicit solutions of the critical types described in that theorem. Conclusions of Theorem 8 and its corollaries are illustrated by this example.

*Remarks.* This paper suggests further study, especially with respect to specialization of the function $z$ which makes the point $p$ a nonunique point for all systems in $H(z)$. Beyond this, a determination of the class of functions $z$ for which the numbers: $M_0, \ldots, M_n$, $R_0, \ldots, R_n$ are invariant is needed. In particular, the case where each component of $z$ is a definitely decreasing function is being studied.

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